The general problem of parametrization
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1. INTRODUCTION

Numerical weather prediction is generally performed by numerical integration of the hydrodynamic equations governing atmospheric motions. Therefore, the differential equations taking a grid-point model, for example, are approximated by finite difference equations applied to a grid of finite volumes. In contrast to the original differential equations which describe the whole spectrum of atmospheric motions, the finite difference equations of a grid-point model describe only those scales which are larger than twice the grid length. For practical reasons the grid length in numerical forecast models cannot be reduced very much below 100 km and, therefore, atmospheric processes on scales smaller than 100 km are excluded from those models. However, small-scale flow affects the mean flow as, for instance, considerable amount of water vapour, sensible heat and momentum are transported by turbulent and convective motions. The effects of the subgrid-scale flow on the mean flow may be ignored for short forecast periods of up to 1 to 2 days, but they become increasingly important for longer periods and must be considered in models for medium-range forecasts and in general-circulation models. Since subgrid-scale processes are not included in models, only their statistical effects on the mean flow can be taken into account. The statistical contributions by the different processes must, therefore, be expressed in terms of the large-scale parameters themselves. The mathematical procedure involved is generally called parametrization.

In the following section the problem of parameterization is discussed from a general point of view, i.e. in relation to the scales of atmospheric motions.

2. THE SPECTRUM OF ATMOSPHERIC MOTIONS

Atmospheric processes are generally observed over a broad spectrum, ranging from microturbulent flow to planetary waves. Fig. 1 gives an idea of the characteristic time scales and length scales of several types of atmospheric process. We notice that microturbulent processes have a characteristic length scale of 1 m, cumulus convection 1 km, deep convection 10 km, mesoscale processes (like tropical cloud clusters) 100 km and synoptical disturbances 1000 km to 10000 km. In addition, Fig. 2 shows how the energy is distributed spectrally near the surface. The spectrum shown is the classical spectrum of horizontal wind speed given by van der Hoven (1957). The spectrum $s(f)$ has a maximum at high frequencies ($f \sim 50$ hours^{-1}) which corresponds to microturbulent flow of length scales of 1 m to 100 m ($f = 1/\tau$ is the frequency, τ is the period of oscillation, $s(f)$ is the spectral energy density). An-other maximum is found for very long periods ($\tau = 4$ days) which reflects synoptical disturbances. A third weaker maximum appears at a period of $\tau = 12$ h which is that of diurnal oscillations. We also observe a broad interval of small values around a period of $\tau = 30$ min with a corresponding length scale of $L = 10$ km ($l = u \cdot \tau$) The smallest scales resolved in a forecast model ($L \sim 100$ km) fall into this spectral interval, so that the spectral region around the first energy maximum belongs entirely to the subgrid-scale. For forecast models the spectral interval of the large-scale disturbances is of primary interest. There have been many attempts in the past to derive the spectral distribution of kinetic energy from observational data, most recently by Chen and Wiin-Nielsen (1978). The investigations show that the kinetic energy follows closely a –3 power distribution for large wave numbers (Fig. 3). The –3 power law seems to be due to the two-dimensional character of the large-scale flow. Three-dimensional isotropic flow which is typical for the small-scale turbulent processes on the other hand shows a –5/3 power distribution. Both distributions are valid only for inertial subranges of the spectrum, where kinetic energy (or enstrophy) is merely transferred from the larger scales (where production occurs) to the smaller scales of dissipation. Theoretical aspects related to this problem are reviewed by Lilly (1973).

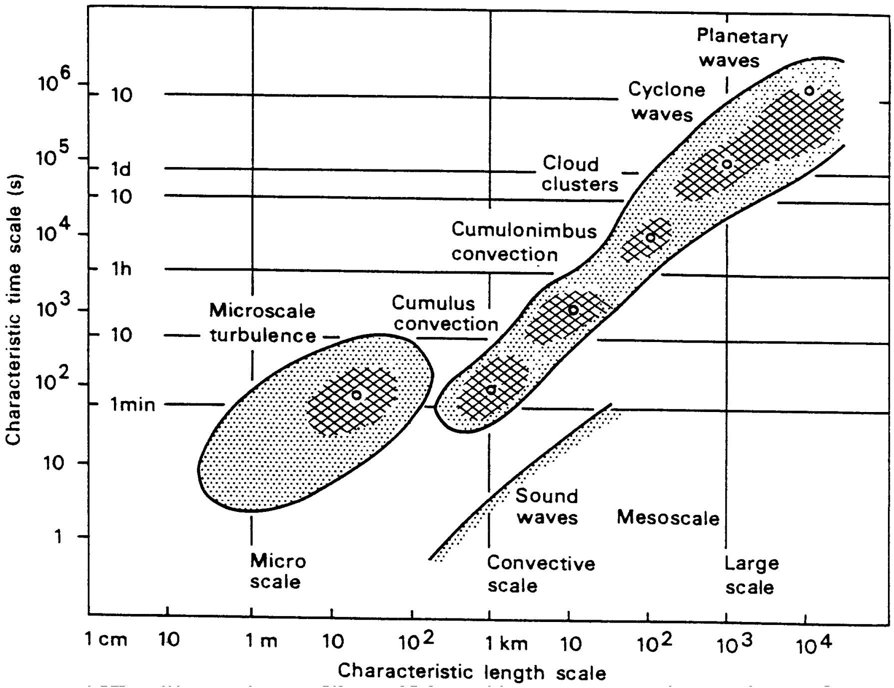

Figure 1. Characteristic scales of atmospheric processes

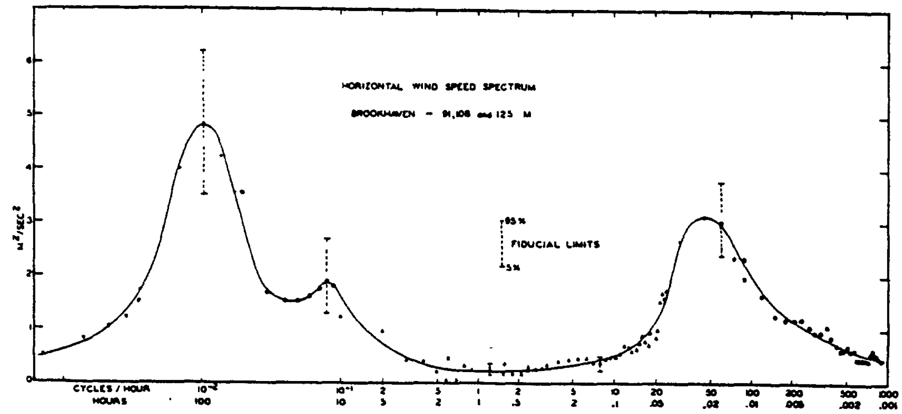

Figure 2. Spectrum of the horizontal wind velocity. After van der Hoven (1957). Some experimental points are shown on the graph.

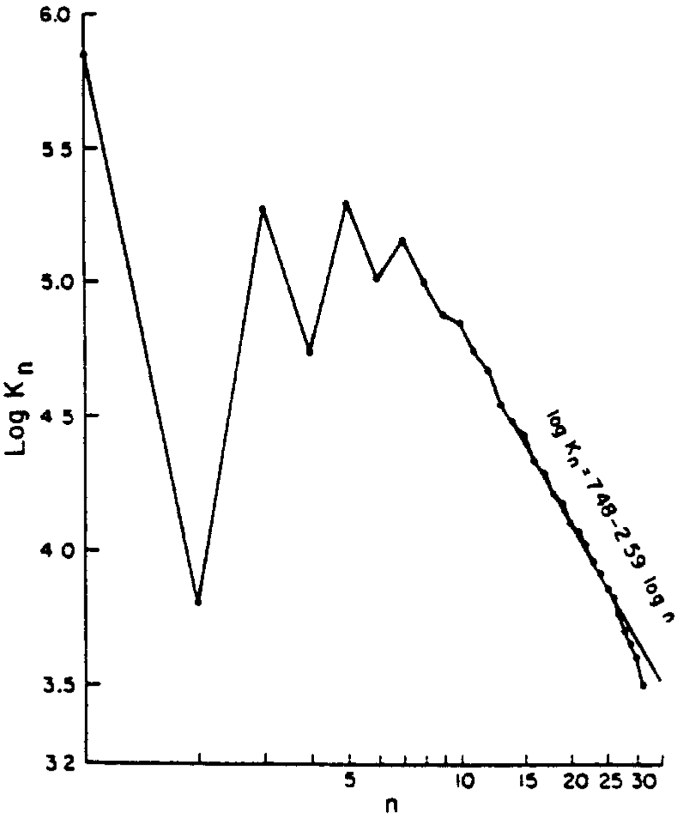

Figure 3. The kinetic energy K for the total atmosphere as a function of a two-dimensional spectral index (n) plotted on logarithmic scales. (After Chen and Wiin-Nielsen (1978)).

3. THE NON-PARAMETRIZED EQUATIONS

In a forecast model only the large-scale flow can be explicitly prescribed. The differential equations of motion must, therefore, be rewritten in such a way that the time evolution of the mean flow, as resolved by the grid, is prescribed. This is achieved by averaging the equations.

For simplicity we consider here processes in dry air. The equations of motion are then

Navier-Stokes equations of motion

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\cdot\mathbf{v}) = -\nabla p - \rho\nabla\varphi - 2\rho\Omega\times\mathbf{v} + \mathbf{F} \qquad (1)$$

Mass continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad (2)$$

First law of thermodynamics

$$\frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e\mathbf{v}) = p\nabla\cdot\mathbf{v} + \rho Q + \varepsilon \qquad (3)$$

Equation of state

$$p = R\rho T \qquad (4)$$

here

p = pressure
$\mathbf{v}$ = velocity
ρ = density of air
$\alpha = \frac{1}{\rho}$ = specific volume
φ = gravitational potential
$\mathbf{F}$ = friction
$e = c_v T$ = internal energy
Q = rate of accession of heat from external sources
ε = rate of conversion of kinetic energy into heat by friction

The frictional force $\mathbf{F}$ results from a convergence of viscous momentum flux as

$$\mathbf{F} = \nabla\cdot P \qquad (5)$$

where P is the stress tensor with the components

$$P_{ij} = \mu\left[\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k} - \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

μ is the coefficient of viscosity, δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq 0$ and $\delta_{ij} = 1$, if $i = j$) and quantities involving a repeated index are to be summed over the index.

The rate of conversion from kinetic energy into internal energy due to viscosity is

$$\varepsilon = \mathbf{P} \cdot \cdot \nabla \mathbf{v} \qquad (6)$$

(the two dots indicate a double scalar product).

The equation for the kinetic energy $\rho k = \frac{1}{2}\rho \mathbf{v}^2$, the equation for the potential energy $\rho\varphi$ and the equation for the internal energy ρe follow from (1) to (5)as

$$\begin{aligned}
\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho \mathbf{v} k + p\mathbf{v} - \mathbf{P} \cdot \mathbf{v}) &= -\rho \mathbf{v} \cdot \nabla\varphi + p\nabla \cdot \mathbf{v} - \mathbf{P} \cdot \cdot \nabla \mathbf{v} \\
\frac{\partial(\rho\varphi)}{\partial t} + \nabla \cdot (\rho \mathbf{v}\varphi) &= \rho \mathbf{v} \cdot \nabla\varphi \\
\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho \mathbf{v} e) &= -p\nabla \cdot \mathbf{v} + \mathbf{P} \cdot \cdot \nabla \mathbf{v} + \rho Q
\end{aligned} \qquad (7)$$

From these equations we see that time changes of the different kinds of energy result from

1) Convergence of energy fluxes across boundaries

$$\begin{aligned}
&\nabla \cdot (\rho \mathbf{v} k + p\mathbf{v} - \mathbf{P} \cdot \mathbf{v}) \\
&\nabla \cdot (\rho \mathbf{v}\varphi) \\
&\nabla \cdot (\rho \mathbf{v} e)
\end{aligned}$$

2) Conversions between the different kinds of energy

$$\begin{aligned}
&\rho \mathbf{v} \cdot \nabla\varphi \\
&p\nabla \cdot \mathbf{v} \\
&\xi = \mathbf{P} \cdot \cdot \nabla \mathbf{v}
\end{aligned}$$

3) External heating
Q (radiative exchanges and heat conduction through the boundaries of a unit volume)

Dissipation of eddy kinetic energy into heat by viscous flow takes place at the smallest eddies of the microturbulent spectral subrange. Their dimensions range from 1 mm to several cm and their time periods are fractions of a second. The viscous flow occurs at the far end of the spectrum shown in Fig. 2 , with $f \gg 1000$.

The prediction equations of a forecast model are derived by averaging the equations (1) to (5). The averaging generally applied is the Reynold's averaging which, for one dimension, takes the form

$$\bar{A}(x) = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} A(x + x')\mathrm{d}x' \qquad (8)$$

The original value A is then defined as the sum of the averaged value $\bar{A}$ (mean value) and a fluctuation A' (or eddy value)

$$A = \bar{A} + A', \quad \text{where} \quad \bar{A'} = 0 \qquad (9)$$

It is convenient to introduce also a weighted operator

$$\widehat{A} = \frac{1}{\bar{\rho}}\overline{\rho A} \tag{10}$$

and we similarly have

$$A = \widehat{A} + A'', \quad \text{where} \quad \widehat{A''} = \overline{\rho A''} \tag{11}$$

To derive the equations for the mean motion we make use of the following rule

$$\overline{\rho XY} = \bar{\rho}\widehat{X}\widehat{Y} + \overline{\rho X''Y''} \tag{12}$$

The equations for the mean motion follow then as

Momentum equation

$$\frac{\partial(\bar{\rho}\hat{\mathbf{v}})}{\partial t} + \nabla\cdot(\bar{\rho}\hat{\mathbf{v}}\hat{\mathbf{v}} + \overline{\rho\mathbf{v}''\mathbf{v}''} - \bar{p}) = -\nabla\bar{p} - \bar{\rho}\nabla\varphi - 2\Omega\times\bar{\rho}\hat{\mathbf{v}} \tag{13}$$

Continuity equation

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot(\bar{\rho}\hat{\mathbf{v}}) = 0 \tag{14}$$

***First law of thermodynamic*s**

$$\frac{\partial(\bar{\rho}\hat{e})}{\partial t} + \nabla\cdot(\bar{\rho}\hat{e}\hat{\mathbf{v}} + \overline{\rho e''\mathbf{v}''} - \bar{p}) = -\bar{p}\nabla\cdot\hat{\mathbf{v}} - \overline{p\nabla\cdot\mathbf{v}''} + \bar{\rho}\hat{\mathbf{Q}} + \overline{\mathbf{P}\cdot\ \cdot\nabla\mathbf{v}} \tag{15}$$

Equation of state

$$\bar{p} = R\bar{\rho}\hat{T} \tag{16}$$

We see that:

The equations (13) to (15) for the mean motion $\hat{\mathbf{v}}$, $\bar{\rho}$ and $\hat{e}$ have the same form as the original equations (1) to (3) for $\mathbf{v}$, ρ, and e, and follow simply from those by replacing the variables by the mean values. However, the equations for the mean value contain additional terms

$$\nabla\cdot(\overline{\rho\mathbf{v}''\mathbf{v}''})$$
$$\nabla\cdot(\overline{\rho e''\mathbf{v}''})$$
$$\overline{p\nabla\cdot\mathbf{v}''}$$

which depend on the eddy motion. The term $-\overline{\rho\mathbf{v}''\mathbf{v}''}$ in the momentum equation is called the Reynold's stress and acts as an additional friction to the average Navier–Stokes stress tensor $\overline{\mathbf{P}}$. From the equations (13) to (15) we can derive the equations for the different kinds of energy. As the kinetic energy splits up into two parts, i.e. those of the mean motion and of the eddy motion

$$\overline{\rho\frac{\mathbf{v}^2}{2}} = \bar{\rho}\frac{\hat{\mathbf{v}}^2}{2} + \frac{1}{2}\overline{\rho\mathbf{v}''^2}$$

the equation for the kinetic energy of the eddy motion must also be considered. The energy equations may be written as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\bar{\rho}\frac{\widehat{\mathbf{v}}^2}{2}\right)+\nabla\cdot\left(\bar{\rho}\,\widehat{\mathbf{v}}\frac{\widehat{\mathbf{v}}^2}{2}+\widehat{\mathbf{v}}\,\bar{p}-\widehat{\mathbf{v}}\,(\bar{p}-\overline{\rho\mathbf{v}''\mathbf{v}''})\right) &&= -\bar{\rho}\,\widehat{\mathbf{v}}\cdot\nabla\varphi+\bar{p}\nabla\cdot\mathbf{v}-\mathbf{P}..\nabla\mathbf{v}+\overline{\rho\mathbf{v}''\mathbf{v}''}..\nabla\mathbf{v}\\
&\frac{\partial}{\partial t}(\bar{\rho}\varphi)+\nabla\cdot(\bar{\rho}\mathbf{v}\varphi) &&= \bar{\rho}\,\widehat{\mathbf{v}}\cdot\nabla\varphi\\
&\frac{\partial}{\partial t}(\bar{\rho}\,\widehat{e})+\nabla\cdot(\bar{\rho}\,\widehat{\mathbf{v}}\,\widehat{e}+\overline{\rho\mathbf{v}''e''+p\mathbf{v}''}) &&= -\bar{p}\nabla\cdot\mathbf{v}+\mathbf{P}..\nabla\mathbf{v}+\mathbf{P}..\nabla\mathbf{v}''+\overline{\mathbf{v}''\cdot\nabla p}\\
&\frac{\partial}{\partial t}\left(\bar{\rho}\frac{\widehat{\mathbf{v}}^2}{2}\right)+\nabla\cdot\left(\overline{\widehat{\mathbf{v}}\rho\frac{\mathbf{v}''^2}{2}}+\overline{\mathbf{v}''\rho\frac{\mathbf{v}''^2}{2}}-\overline{\mathbf{v}''\mathbf{P}}\right) &&= -\overline{\rho\mathbf{v}''\mathbf{v}''}..\nabla\mathbf{v}-\mathbf{P}..\nabla\mathbf{v}''-\overline{\mathbf{v}''\cdot\nabla p}
\end{aligned}
\qquad (17)
$$

As was done for the energy equations (7) for the non-averaged flow, the energy conversions between the different kinds of energies can be seen from the equations (17) for the mean flow and for the eddy flow.

In contrast to the differential case, there are now explicit conversion terms from the kinetic energy of the mean flow to that of the eddy flow and from the eddy kinetic energy to the internal energy.

1) Dissipative heating due to viscous momentum flux.

2) Generation of turbulent kinetic energy by mechanical turbulence in a shear flow.

3) The work done by the turbulent flow against the pressure gradient. The largest contribution comes from the vertical component in the boundary layer as the pressure gradient term is related to the turbulent heat flux

$$-\overline{w''\frac{\partial p'}{\partial z}} \sim g\overline{w''\rho'} \sim \frac{g}{\bar{\theta}}\bar{\rho}\overline{w''\theta'}$$

Thus turbulent kinetic energy is generated/destroyed depending on the turbulent heat flux being upward or downward.

4. Principles of parametrization

Considering the prognostic equations (13) to (15) for the mean motion, we are faced with the general problem of parametrization, i.e. to determine the terms depending on viscosity and on the eddy motion. The problem is generally solved in forecast models as follows:

The viscous terms are not taken into account, as the sink of kinetic energy of the mean flow due to viscous flow is small compared with that resulting from the transfer to eddy kinetic energy. Consequently, the warming effect due to dissipation is also neglected in the thermodynamic equation.

The eddy terms are generally specified in one of the following three ways.

1) Eddy fluxes are neglected.

2) Eddy fluxes are determined by means of the K-theory.

3) Eddy fluxes are specified using higher order closure schemes.

The first scheme is often used in barotropic models. Neglect of the eddy terms in barotropic models seems justified, since the energetic conversions in barotropic flow are reduced to transformations between different kinds of mechanical energy. Horizontal diffusion is, however, sometimes included for reason of numerical stability.

The K-approach is widely adopted in baroclinic numerical forecast models. This method is based on the assumption that the eddy flow yields down-gradient transfer of momentum and sensible heat. Taking the eddy momentum flux for example, we would have

$$\overline{\rho \mathbf{v}''\mathbf{v}''} = -\bar{\rho} K_{\mathbf{M}} \nabla \widehat{\mathbf{v}}$$

where $K_{\mathbf{M}}$ is the diffusion coefficient of momentum. The K-theory is applied to determine eddy fluxes in the horizontal direction as well as in the vertical direction. An example of the K-approach is given in the lectures on the parameterization of vertical eddy fluxes in the planetary boundary layer.

Higher-order closure schemes use prediction equations for the eddy fluxes $\overline{\rho u''x''}$. These equations are similar to the eddy kinetic equation and contain triple products of eddy variables. These triple products must be specified either in terms of the predicted values (parametrized) or again be predicted, which leaves the problem of parametrization at a higher level. A hierarchy of higher-order closure models for the planetary boundary layer have been given by Mellor and Yamada (1974).

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